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Information-Entropy and Purity of Decoherence Functions

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The structure of the spaces of propositions and decoherence functions in the consistent-histories approach to generalized quantum theory is outlined. It is shown that although the space of decoherence functions is convex, there are no pure decoherence functions. A definition of information-entropy is described which no needs no *a priori* notion of time.

1. INTRODUCTION

A particularly attractive feature of the consistent-histories program, as developed by Gell-Mann and Hartle (see, for example, Hartle, 1993), following pioneering work by Griffiths (1984) and Omnès (1992), is that it offers a framework for quantum theory in which time potentially plays a subsidiary role. The central idea of the scheme is that under certain *consistency* conditions it is possible to assign probabilities to generalized histories of a system. In normal quantum theory such histories are represented by time-ordered strings of propositions; however, the scheme allows for much more general histories in which there is no *a priori* notion of time ordering. These generalized histories are expected to play a key role in application of the formalism to quantum gravity.

In the generalized version of the history scheme developed in Isham (1994), Isham and Linden (1994), and Isham *et al*. (1994) the central mathematical ingredients are a set of histories $\mathcal{UP}$ (or, more accurately, the set of *propositions* about histories) and an associated set of decoherence functions $\mathcal{D}$, with the pair $(\mathcal{UP}, \mathcal{D})$ being regarded as the analogue in the history theory

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of the pair ($\mathcal{L}$, $\mathcal{S}$) in standard quantum theory, where $\mathcal{L}$ is the lattice of propositions and $\mathcal{S}$ is the space of states on $\mathcal{L}$.

2. THE CONVEX SET OF DECOHERENCE FUNCTIONS

Isham (1994) and Isham and Linden (1994) describe how the space $\mathcal{UP}$ encodes the generalized quantum temporal logic of the propositions. As explained there, there are compelling reasons for postulating that the natural mathematical structure on $\mathcal{UP}$ is that of an *orthoalgebra* (Foulis *et al.*, 1992) with the three orthoalgebra operations $\oplus$, $\neg$, and $<$ corresponding, respectively, to the disjoint sum, negation, and coarse-graining operations involved by Gell-Mann and Hartle. One example of an orthoalgebra is the lattice of projection operators on a Hibert space. In this case, the operation $\oplus$ is defined on disjoint pairs of projectors P, Q with $P \oplus Q = P \vee Q$, where, as usual, $P \vee Q$ denotes the projector onto the linear span of the subspaces onto which P and Q project. In the example of a lattice, $\vee$ is defined on all projectors not only on pairs that are disjoint.

Throughout this article I will consider the case where the orthoalgebra of propositions is the space of projectors on a Hilbert space $\mathcal{V}$, which, for the sake of simplicity, will be taken to be finite-dimensional. This Hilbert space may arise from having propositions at n time points, in which case $\mathcal{V} = \otimes^n \mathcal{H}$ (see below), but it need not do so. A crucial ingredient in the construction of the information-entropy will be the *dimension* of a proposition, defined to be the dimension of the projector that represents the proposition on $\mathcal{V}$.

One important motivation for this framework is the fact that discrete-time histories in quantum theory can indeed be given the structure of an orthoalgebra. The key idea is that an n-time, homogeneous history proposition $(\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n})$ can be associated with the operator $\alpha_{t_1} \otimes \alpha_{t_2} \otimes \ldots \otimes \alpha_{t_n}$ which is a genuine *projection* operator on the n-fold tensor product $\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2} \otimes \ldots \otimes \mathcal{H}_{t_n}$ of n-copies of the Hilbert-space $\mathcal{H}$ on which the canonical theory is defined (Isham, 1994; Isham and Linden, 1994).

As an example of how this tensor product structure encodes the temporal logic, consider the case of propositions at two times and the homogeneous history proposition 'α and then β' (represented by the projection operator $\alpha \otimes \beta$ on $\mathcal{H} \otimes \mathcal{H}$). The negation of this proposition is represented by $1 \otimes 1 - \alpha \otimes \beta$, which may be rewritten

$$1 \otimes 1 - \alpha \otimes \beta = ((1 - \alpha) \otimes \beta) + (\alpha \otimes (1 - \beta)) \\ + ((1 - \alpha) \otimes (1 - \beta))$$

which precisely encodes the fact that 'not (α and then β)' is equivalent to the proposition '(not α and then β) or (α and then not β) or (not α and then not β).' The tensor product structure of temporal propositions in this case also suggests the intriguing possibility of temporal entanglement (Isham and Linden, 1995).

Isham and Linden (1994) argue that the properties of the decoherence function d: $\mathcal{UP} \times \mathcal{UP} \to \mathbb{C}$ are:

(a) *Hermiticity*: $d(\alpha, \beta) = d(\beta, \alpha)^*$ for all $\alpha, \beta \in \mathcal{UP}$.

(b) *Positivity*: $d(\alpha, \alpha) \geq 0$ for all $\alpha \in \mathcal{UP}$.

(c) *Additivity*: if α and β are disjoint, then, for all γ, $d(\alpha \oplus \beta, \gamma) = d(\alpha, \gamma) + d(\beta, \gamma)$.

(d) *Normalization*: $d(1, 1) = 1$.

In the case that $\mathcal{UP}$ is the lattice of projectors on a finite-dimensional Hilbert space $\mathcal{V}$ (not necessarily arising as a temporal tensor product), it is possible to classify all decoherence functions satisfying these properties as follows (Isham *et al.*, 1994).[2] Decoherence functions are in one-to-one correspondence with 'decoherence operators' X *on* $\mathcal{V} \otimes \mathcal{V}$ according to the rule

$$d(\alpha, \beta) = \mathrm{tr}_{\mathcal{V}\otimes\mathcal{V}}(\alpha \otimes \beta\ X) \tag{1}$$

where the decoherence operator X satisfies (a) $MXM = X^{\dagger}$, where $M(u \otimes v) := v \otimes u$; (b) $\mathrm{tr}_{\mathcal{V}\otimes\mathcal{V}}(\alpha \otimes \alpha X) \geq 0$; (c) $\mathrm{tr}_{\mathcal{V}\otimes\mathcal{V}}(X) = 1$. X need not be a positive operator. Indeed, Isham and Linden (1994) give examples of decoherence functions in standard quantum theory where $d(\alpha, \alpha) > d(\beta, \beta)$ for two histories α and β for which $\alpha \leq \beta$, and we also found decoherence functions and histories γ for which $d(\gamma, \gamma) > 1$.

It may be noted that if d_1 and d_2 are decoherence functions, then so is

$$d_{(\lambda)} := \lambda d_1 + (1 - \lambda) d_2 \tag{2}$$

where λ is a real constant, $0 \leq \lambda \leq 1$. Thus the space of decoherence functions is a convex set. However, if X is a decoherence operator, then so are $X + Y$ and $X - Y$, where $Y = i\,(s_1 \otimes s_2 - s_2 \otimes s_1)$ for any self-adjoint operators s_1, s_2 on $\mathcal{V}$. Thus we may write the decoherence function d_X associated to X as $d_X \equiv 1/2 d_{(X+Y)} + 1/2 d_{(X-Y)}$. The decoherence functions $d_{(X+Y)}$ and $d_{(X-Y)}$ associated to the decoherence operators $X + Y$ and $X - Y$ are different from d_X in general. Thus any decoherence function may be written as a convex sum and so there are no pure decoherence functions.

3. INFORMATION-ENTROPY

I turn now to the question of defining the information-entropy in the context of a window and for a given decoherence function (Isham and Linden,

[2] Generalizations of this result have been given by Wright (1995) and Rudolph (1996) and other aspects of the structure of the space of decoherence functions are described by Schreckenberg (1997).

1997). What we seek is a notion of information-entropy that can be used in generalized history quantum theory. In particular, the definition should be applicable in principle to systems in which the concept of time is not fundamental and may emerge only in some coarse-grained way. Furthermore, even if the system has a standard notion of time, the information-entropy—which encodes the number of bits required to describe the system—may not necessarily all reside in the initial state. The description of a system in this generalized type of quantum theory is given entirely in terms of the set of propositions and the values of the decoherence function, so we must construct our measure of information-entropy solely from these.

Each consistent set or "window" gives a probability distribution $\text{Prob}(\alpha_i) = d(\alpha_i, \alpha_i)$ for the histories $\{\alpha_i\}_{i=1}^{N}$. Thus, a simple first attempt at such a definition might be $I_{\text{trial}} = -\sum_{i=1}^{n} d(\alpha_i, \alpha_i) \log d(\alpha_i, \alpha_i)$. However, this does not decrease under refinement of the consistent set; (indeed the coarsest possible window $\{0, 1\}$, where 1 is the history which is always realized, has the minimum value (zero) of I_{trial}.

In a very interesting paper, Hartle (1995), proposes an approach to these issues using maximum entropy ideas, and this paper was part of the motivation for what follows [see also Gell-Mann and Hartle, (1995), Halliwell (1993), and Kent (1996) for other discussions of information-entropy in the histories approach]. However, it is possible to arrive at a simple definition of information-entropy in generalized quantum mechanics more directly. Specifically, Isham and Linden (1997) put forward the following definition of the information-entropy for a decoherence functon and window:

$$I_{d,W} := -\sum_{i=1}^{N} d(\alpha_i, \alpha_i) \log \frac{d(\alpha_i, \alpha_i)}{(\dim \alpha_i / \dim \mathcal{V})^2} \tag{3}$$

The considerations leading to (3) are described in Isham and Linden (1997) and a number of properties developed. In particular, it is shown that it has the key property that it decreases under refinement of the consistent set.

A natural possibility is to define *the* information-entropy of the decoherence function *d as the minimum over all consistent sets of* $I_{d,W}$, i.e.,

$$I_d := \min_{W} I_{d,W} \tag{4}$$

In examples it is found that the consistent set (or sets) which minimize $I_{d,W}$ are naturally associated with the decoherence operator X. It is also shown how in the case of the history version of standard quantum mechanics with unitary time evolution, the value of the information-entropy defined by (4) is (up to normalization) $-\text{tr}(\rho \log \rho)$ for the case when histories containing homogeneous projectors are considered; thus all the information-entropy lies in the initial density matrix in this case.

It is worth noting that fundamental to the consistent-histories approach from the start has been the idea of taking the sum of two homogeneous histories in standard n-time quantum theory to form inhomogeneous histories. However, the idea of the dimension of an inhomogeneous history is difficult to understand unless, as Isham and I have frequently advocated, histories are identified with projection operators on an n-fold tensor product space.

I have called the function $I_{d,W}$ a measure of information-entropy for generalized quantum mechanics as it has key properties that it descreases under refinement and it is small for consistent windows in which the probability is peaked around histories of small dimensions. While, for reasons described in Isham and Linden (1997), I have taken (3) as the measure of information-entropy, it should be noted that any function of the form

$$I^x_{d,W} = -\sum_{i=1}^{N} d(\alpha_i, \alpha_i) \log \left[\frac{d(\alpha_i, \alpha_i)}{(\dim \alpha_i / \dim \mathcal{V})^x} \right] \tag{5}$$

where $x \geq 1$ is a real number, also has the key property that it decreases under refinement of the consistent set. The case $x = 1$ may turn out to be the most interesting, as in this case the measure of information is (minus) the Kullback information of the distribution $\{d(\alpha_i, \alpha_i)\}$ relative to a 'maximally ignorant' distribution on the set $\{\alpha_i\}$ which has $\text{Prob}(\alpha_i) = \dim(\alpha_i)/\dim \mathcal{V}$. Interestingly, Gell-Mann and Hartle (1995) have considered measures of this sort as a result of rather different considerations, such as the idea of thermodynamic depth (Lloyd and Pagels, 1988). The relationship between the measures with different values of x needs to be understaood.

I anticipate that this definition of information-entropy—which is a straightforward function on the class of consistent sets with attractive properties under refinement—may help in the development of a set selection criterion[3]: for example, in the case that the system naturally divides into a subsystem and the 'environment' this might be done by selecting the set which minimizes the information-entropy of the distinguished subsystem (see, for example, Zurek, 1994). In this context, it should be noted that if our vector space $\mathcal{V}$ happens to arise as the tensor product of two spaces $\mathcal{V}_1$ and $\mathcal{V}_2$, then the definition of information-entropy has precisely the behavior that might be hoped for; for if one considers a consistent window in which each history proposition α is a tensor product $\alpha = \alpha_1 \otimes \alpha_2$, with $\alpha_1 \in P(\mathcal{V}_1)$ and $\alpha_2 \in P(\mathcal{V}_2)$, then the information-entropy is the sum of the information-entropy associated to each subsystem.

[3] The importance of this issue for the whole framework has been discussed by Gell-Mann and Hartle (1995) and Dowker and Kent (1995).

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